

Laurent Expansions for Vertex Operators

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Abstract

A method is presented for using coherent vectors to calculate the explicit form of Schur polynomials which are the coefficients of Laurent expansion of a vertex operator.

1 Preliminaries

Let $\Gamma_0\mathcal{D}$ be a Bose algebra (cf. [4]) i.e. a commutative graded algebra generated by a pre-Hilbert space $\mathcal{D}, \langle \cdot, \cdot \rangle$ (the so-called one-particle space) and the unity ϕ (the vacuum) provided with the extension $\langle \cdot, \cdot \rangle$ of the scalar product of \mathcal{D} making ϕ a unit vector and fulfilling the property that for every $x \in \mathcal{D}$, the adjoint x^* to the operator of multiplication by x is defined on the whole $\Gamma_0\mathcal{D}$ and constitutes a derivation (i.e. fulfils the Leibniz rule). We make the space $\tilde{\Gamma}\mathcal{D}$ of all antilinear functionals on $\Gamma_0\mathcal{D}$ the extension of $\Gamma_0\mathcal{D}$ by identifying $f \in \Gamma_0\mathcal{D}$ with the antilinear functional $\langle \cdot, f \rangle$. The space $\tilde{\Gamma}\mathcal{D}$ can be naturally made into an algebra containing $\Gamma_0\mathcal{D}$ as a subalgebra. We consider $\tilde{\Gamma}\mathcal{D}$ as a locally convex space with the weak topology $\sigma(\tilde{\Gamma}\mathcal{D}, \Gamma_0\mathcal{D})$. The weak closure $\tilde{\mathcal{D}}$ of \mathcal{D} is a subspace of $\tilde{\Gamma}\mathcal{D}$. It is easy to show that $\Gamma_0\mathcal{D}, \langle \cdot, \cdot \rangle$ admits the completion $\Gamma\overline{\mathcal{D}}$ within $\tilde{\Gamma}\mathcal{D}$.

We shall use the exponentials of elements $w \in \mathcal{D}$,

$$e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \in \Gamma\overline{\mathcal{D}},$$

which are called *coherent vectors*. In [4] the following relations are verified:

$$\langle a, b \rangle^j = \frac{1}{j!} \langle a^j, b^j \rangle \quad (1)$$

$$(x^n)^* e^w = \langle x, w \rangle^n e^w \quad (2)$$

$$\langle e^u, fg \rangle = \langle e^u, f \rangle \langle e^u, g \rangle \quad (3)$$

$$e^{\mathbf{a}(w)} e^v = e^{\langle w, v \rangle} e^v. \quad (4)$$

Also a proof that the set $\{e^x : x \in \mathcal{D}\}$ of coherent vectors is total in $\Gamma \overline{\mathcal{D}}$ can be found in [4].

2 The Laurent Expansion for a Vertex operator

Let \mathcal{D} be spanned by an orthonormal system $\{f_n\}$ and by an orthonormal system $\{g_n\}$ as well. The operator valued functions of z

$$V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g_n^*} : \Gamma_0 \mathcal{D} \rightarrow \tilde{\Gamma} \mathcal{D},$$

shall be called a *vertex operator* (cf.[2],[3],[1]).

Write (\mathbf{p}, \mathbf{q}) for tuples of non-negative integers

$$(\mathbf{p}, \mathbf{q}) = (p_1, q_1, p_2, q_2, \dots, p_k, q_k, \dots)$$

and define

$$\mathfrak{N}_m = \left\{ (\mathbf{p}, \mathbf{q}) : \sum_{k=1}^{\infty} (p_k + q_k) = m \right\}$$

and

$$\mathfrak{N}^w = \left\{ (\mathbf{p}, \mathbf{q}) : \sum_{k=1}^{\infty} (p_k + q_k) < \infty, \sum_{j=1}^{\infty} j(p_j - q_j) = w \right\}.$$

For $\mathfrak{s} = (s_1, s_2, \dots)$, write

$$\mathfrak{s}! = \prod_{k=1}^{\infty} s_k!.$$

We prove the following

THEOREM *Vertex operators admit the weak evaluation on $\Gamma_0\mathcal{D}$ and the weak convergent Laurent expansion*

$$V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g_n^*} = \sum_{w \in \mathbb{Z}} \mathcal{S}_w \{f_n, g_n^*\} z^w$$

with coefficients

$$\mathcal{S}_w \{f_n, g_n^*\} = \sum_{m=0}^{\infty} \sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}_m \cap \mathfrak{N}^w} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^*$$

called the Schur polynomials (cf.[3]).

To prove the Theorem we shall need the following

LEMMA *Take any pair of elements $u, v \in \mathcal{D}$. Then the element $V(z) e^u$ is well defined in $\tilde{\Gamma}\mathcal{D}$ and we have*

$$\langle e^u, V(z) e^v \rangle = \left\langle e^u, \left(\sum_{w \in \mathbb{Z}} \mathcal{S}_w \{f_n, g_n^*\} z^w \right) e^v \right\rangle, \quad (5)$$

where

$$\mathcal{S}_w \{f_n, g_n^*\} = \sum_{m=0}^{\infty} \sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}_m \cap \mathfrak{N}^w} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^*.$$

Proof. . Take $u, v \in \mathcal{D}$. By virtue of (4) we get

$$\langle e^u, e^{\mathbf{a}^+(x)} e^{\mathbf{a}^-(y)} e^v \rangle = e^{\langle u, v \rangle} e^{\langle u, x \rangle + \langle y, v \rangle},$$

and consequently

$$\langle e^u, V(z) e^v \rangle = e^{\langle u, v \rangle} e^{\sum_{n=1}^{\infty} (\langle f_n, u \rangle z^n + \langle v, g_n \rangle z^{-n})}.$$

Since u and v are linear combinations of f_k and g_k respectively, $\langle f_n, u \rangle z^n = \langle v, g_n \rangle z^{-n} = 0$ for large n . Due to (2) we get

$$\begin{aligned} & \left\langle e^u, \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^* e^v \right\rangle \\ &= \left\langle \left(\prod_{k=1}^{\infty} f_k^{p_k} \right)^* e^u, \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^* e^v \right\rangle = \left(\prod_{k=1}^{\infty} \langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k} \right) e^{\langle u, v \rangle}, \end{aligned}$$

where all the products are finite and they are non-zero only when p_k and q_k are zeros for f_k and g_k orthogonal to v and u respectively. Consequently

$$\begin{aligned}
& \frac{1}{m!} \left(\sum_{n=1}^{\infty} \langle f_n, u \rangle z^n + \sum_{n=1}^{\infty} \langle v, g_n \rangle z^{-n} \right)^m \\
&= \sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}_m} \frac{1}{\mathbf{p}! \mathbf{q}!} \prod_{k=1}^{\infty} (\langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k} z^{k(p_k - q_k)}) \\
&= \sum_{w \in \mathbb{Z}} \sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}^w \cap \mathfrak{N}_m} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} \langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k} \right) z^w \\
&\left\langle e^u, \sum_{w \in \mathbb{Z}} \left(\sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}^w \cap \mathfrak{N}_m} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle e^{-\langle u, v \rangle}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{1}{m!} \left(\sum_{n=1}^{\infty} \langle f_n, u \rangle z^n + \sum_{n=1}^{\infty} \langle v, g_n \rangle z^{-n} \right)^m \\
&= \left\langle e^u, \sum_{w \in \mathbb{Z}} \left(\sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}_m \cap \mathfrak{N}^w} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle e^{-\langle u, v \rangle},
\end{aligned}$$

and finally

$$\begin{aligned}
\langle e^u, V(z) e^v \rangle &= e^{\langle u, v \rangle} e^{\sum_{n=1}^{\infty} (\langle f_n, u \rangle z^n + \langle v, g_n \rangle z^{-n})} \\
&= \left\langle e^u, \sum_{m=0}^{\infty} \sum_{w \in \mathbb{Z}} \left(\sum_{(\mathbf{p}, \mathbf{q}) \in \mathfrak{N}_{m,w}} \frac{1}{\mathbf{p}! \mathbf{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k} \right) \left(\prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle
\end{aligned}$$

which concludes the proof of the Lemma.

Proof of the Theorem

Since $\Gamma_0 \mathcal{D}$ is the linear span of the set $\{x^k : x \in \mathcal{D}, k = 1, 2, \dots\}$ (cf.[4]), it is sufficient to show that for any $u, v \in \mathcal{D}$ and any natural numbers k, j we have

$$\langle u^k, V(z) v^j \rangle = \left\langle u^k, \left(\sum_{w \in \mathbb{Z}} \mathcal{S}_w \{f_n, g_n^*\} z^w \right) v^j \right\rangle$$

which follows by differentiating respectively k and j times at 0 the variables t and s of the identity (5) with tu and sv substituted for u and v . ■

References

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